STOCHASTIC WAVE EQUATIONS WITH NONLINEAR DAMPING AND SOURCE TERMS

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ABSTRACT. In this paper, we discuss an initial boundary value problem for the stochastic wave equation involving the nonlinear damping term $|u_t|^{q-2}u_t$ and a source term of the type $|u|^{p-2}u$. We firstly establish the local existence and uniqueness of solution by the Galerkin approximation method and show that the solution is global for $q \geq p$. Secondly, by an appropriate energy inequality, the local solution of the stochastic equations will blow up with positive probability or explosive in energy sense for $p > q$.

1. INTRODUCTION

The wave equation of the following form

$$\begin{cases} u_{tt} - \Delta u + a|u_t|^{q-2}u_t = b|u|^{p-2}u, & (x,t) \in D \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial D \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in D, \end{cases} \tag{1.1}$$

where D is a bounded domain in $\mathbb{R}^d$ with a smooth boundary ∂D, a, $b > 0$ are constants, has been extensively studied and results concerning existence, blow-up and asymptotic behavior of smooth, as well as weak solutions have been established by several authors over the past three decades. For $b = 0$, it is well known that the damping term assures global existence and decay of the solution energy for arbitrary initial data (see [1] and [2]). For $a = 0$, the source term causes finite time blow-up of solutions with large initial data (negative initial energy), see [3] and [4]. The interaction between the damping term $a|u_t|^{q-2}u_t$ and the source term $b|u|^{p-2}u$ makes the problem more interesting. This situation was first considered by Levine [5, 6] in the linear damping case ($q = 2$), where he showed that solutions with negative initial energy blow up in finite time. In [7], Georgiev and Todorova extended Levine's result to the nonlinear damping case ($q > 2$). In their work, the authors introduced a new method and determined relations between q and p for which there is finite time blow-up. Specifically, they showed that solutions with negative energy continue to exist globally in time if $q \geq p \geq 2$ and blow up in finite time if $p > q \geq 2$ and the initial energy is sufficiently negative. Messaoudi [8] extended the blow-up result of [7] to solutions with only negative initial energy. For related results, we refer the reader to Levine and Serrin [9], Levine and Ro Park [10], Vitillaro [11] and Messaoudi and Said-Houari [12].

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In fact, the driving force may be affected by the environment randomly. In view of this, we consider the following stochastic wave equations

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{q-2}u_t = |u|^{p-2}u + \varepsilon\sigma(u, \nabla u, x, t)\partial_t W(t, x), & (x, t) \in D \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial D \times (0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in D, \end{cases} \tag{1.2}$$

where $q \geq 2$, $p > 2$, ε is given positive constant which measures the strength of the noise, and $W(t, x)$ is a Wiener random field, which will be defined precisely later, and the initial data $u_0(x)$ and $u_1(x)$ are given functions.

To motivate our work, let us recall some results regarding stochastic wave equations with linear damping ($q = 2$). For the blow-up results, Chow [13] discussed a class of non-dissipative stochastic wave equations with polynomial nonlinearity in $\mathbb{R}^d$ with $d \leq 3$. Using the energy inequality the author demonstrated the blow-up in finite time with a positive probability or explosive in L^2 norm for an example and studied the global existence of the solutions for the equation. This blow-up result has been later generalized by the same author in [14]. In a recent paper, using the energy inequality, Bo et al. [15] proposed sufficient conditions that the solutions of a class of stochastic wave equations blow up with a positive probability or explosive in L^2 sense. In those papers, the main tool in proving explosive/blow-up is the "concavity method" where the basic idea of the method is to construct a positive defined functional $F(t)$ of the solution by the energy inequality and show that $F^{-\alpha}(t)$ is a concave function of t. Unfortunately, this method fails in the case of a nonlinear damping term ($q > 2$). For the global existence and invariant measure, Chow [16, 17] studied properties of the solution of (1.2) with $q = 2$ such as asymptotic stability and invariant measure and Brzeźniak et al. [18] studied global existence and stability of solutions for the stochastic nonlinear beam equations. There are also many other works on the stochastic wave equations with global existence and invariant measure for linear damping, see references in [19, 20, 21, 22].

Nonlinear stochastic wave equations with nonlinearity on the damping were first studied by Pardoux [23]. But the progress is little in nearly three decades. Recently, J.U. Kim [24] and V. Barbu et al. [25] considered an initial boundary value stochastic wave equations with nonlinear damping and dissipative damping, respectively. They proved the existence of an invariant measure. However, to our knowledge, the explosive/blow-up results with nonlinearity on the damping seems to be studied here for the first time. Since the existence and uniqueness of a solution for the deterministic equation ($\varepsilon = 0$) is well known under some assumptions with nonlinearity on the damping, we may anticipate similar results for the stochastic equation. However, the methods used in earlier works on the stochastic wave equation with linear damping do not work. Hence, we will employ the Galerkin approximation method to establish the local existence and uniqueness solution for (1.2). For multiplicative noise, i.e., when σ depend on u and ∇u, we need to obtain the mean energy estimates, but this is some technical difficulty. This is also a major hurdle for the uniqueness of a solution of (1.2). So here we consider only additive noise, i.e. $\sigma(u, \nabla u, x, t) = \sigma(t, x, \omega)$ so that the stochastic integral may be well defined as an $L^2(D)$-valued continuous martingale. We will prove the global solution of (1.2) for $q \geq p$. Concerning explosive/blow-up results, we use the technique of [7] with a modification in the energy functional due to the different nature of the problems for $p > q$.

This paper is organized as follows. In Section 2 we present some assumptions and definitions needed for our work. In Section 3, we show the local existence and uniqueness solution of (1.2) and prove the solution being global for $q \geq p$. Section 4 is devoted to the proof of the explosive solutions of (1.2) for $p > q$.

2. Preliminaries

Firstly, let us introduce some notation used throughout this paper. We set $H = L^2(D)$ with the inner product and norm denoted by $(\cdot, \cdot)$ and $||\cdot||_2$, respectively. Denote by $||\cdot||_q$ the $L^q(D)$ norm for $1 \leq q \leq \infty$ and by $||\nabla \cdot||_2$ the Dirichlet norm in $V = H_0^1(D)$ which is equivalent to the $H^1(D)$ norm. We also set $q,\ p$ satisfy

$$\begin{cases} q \geq 2, \quad p > 2, \quad \max\{p,\ q\} \leq \dfrac{2(d-1)}{d-2}, & \text{if } d \geq 3, \\ q \geq 2, \quad p > 2, & \text{if } d = 1, 2, \end{cases} \tag{2.1}$$

which implies that $H_0^1(D)$ is continuously compact embedded into $L^p(D)$. Hence, we have the Sobolev inequality

$$||u||_{2(p-1)} \leq c||\nabla u||_2, \quad \forall u \in H_0^1(D), \tag{2.2}$$

where c is the embedding constant of $H_0^1(D) \subseteq L^p(D)$. Using (2.2), we have the following inequality

$$||u^{p-2}v||_2 \leq c^{p-1}||\nabla u||_2^{p-2}||\nabla v||_2, \quad \forall u,\ v \in H_0^1(D). \tag{2.3}$$

In fact, when $d = 1,\ 2$, let $q > 1$ and $k = \frac{q}{q-1}$, by the Hölder inequality and (2.2) we have

$$||u^{p-2}v||_2 \leq ||u||_{2(p-2)q}^{p-2}||v||_{2k} \leq c^{p-1}||\nabla u||_2^{p-2}||\nabla v||_2. \tag{2.4}$$

When $d > 2$, set $q = \frac{d}{(d-2)(p-2)} > 1$. Then $k = \frac{d}{d-(d-2)(p-2)} \leq \frac{d}{d-2}$, (2.4) is also valid for $d > 2$.

Let $(\Omega, P, \mathcal{F})$ be a complete probability space for which a $\{\mathcal{F}_t,\ t \geq 0\}$ of sub-σ-fields of $\mathcal{F}$ is given. A point of Ω will be denoted by ω and $\mathbf{E}(\cdot)$ stands for expectation with respect to probability measure P. When $\mathcal{O}$ is a topological space, $\mathcal{B}$ denotes the Borel σ-algebra over $\mathcal{O}$. Suppose that $\{W(t,x) : t \geq 0\}$ is a V-valued R-Wiener process on the probability space with the variance operator R satisfying $TrR < \infty$. Moreover, we can assume that R has the following form

$$Re_i = \lambda_i e_i, \quad i = 1, 2, \cdots,$$

where λ_i are eigenvalues of R satisfying $\sum_{i=1}^{\infty} \lambda_i < \infty$ and $\{e_i\}$ are the corresponding eigenfunctions with $c_0 := \sup_{i\geq 1} ||e_i||_\infty < \infty$ (where $||\cdot||_\infty$ denotes the super-norm). To simplify the computations, we assume that the covariance operator R and $-\Delta$ with homogeneous Dirichlet boundary condition have a common set of eigenfunctions, i.e., $\{e_i\}_{i=1}^{\infty}$ satisfy

$$\begin{cases} -\Delta e_i = \mu_i e_i, & x \in D, \\ e_i = 0, & x \in \partial D, \end{cases} \tag{2.5}$$

and form an orthonormal base of V. In this case,

$$W(t,x) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} B_i(t) e_i,$$

where $\{B_i(t)\}$ is a sequence of independent copies of standard Brownian motions in one dimension. Let $\mathcal{H}$ be the set of $L_2^0 = L^2(R^{\frac{1}{2}}V, V)$-valued processes with the norm

$$||\Psi(t)||_{\mathcal{H}} = \Big(\mathbf{E}\int_0^t ||\Psi(s)||^2_{L_2^0} ds\Big)^{\frac{1}{2}} = \Big(\mathbf{E}\int_0^t Tr\big(\Psi(s)R\Psi^*(s)\big)ds\Big)^{\frac{1}{2}} < \infty,$$

where $\Psi^*(s)$ denotes the adjoint operator of $\Psi(s)$. Let $\{t_k\}_{k=1}^n$ be a partition on $[0,T]$ such that $0 = t_0 < t_1 < \cdots < t_n = T$. For a process $\Psi(t) \in \mathcal{H}$, define the stochastic integral with respect to the R-Wiener process as

$$\int_0^t \Psi(s)dW(s) = \lim_{n\to\infty}\sum_{k=0}^{n-1}\Psi(t_k)(W(t_{k+1}\wedge t) - W(t_k\wedge t)), \tag{2.6}$$

where the sequence converges in $\mathcal{H}$-sense. It is not difficult to check that the integral process $\int_0^t \Psi(s)dW(s)$ is a martingale for any $\Psi(t) \in \mathcal{H}$, and the quadratic variation process is given by

$$\Big\langle\Big\langle \int_0^t \Psi(s)dW(s)\Big\rangle\Big\rangle = \int_0^t Tr\big(\Psi(s)R\Psi^*(s)\big)ds.$$

For more details about the infinite dimension Wiener process and the stochastic integral, we refer to [26].

Finally, we give the definition of solution to (1.2). For the definition of a solution, we assume that

$$(u_0, u_1) \in H_0^1(D) \times L^2(D), \tag{2.7}$$

and that $\sigma(x,t)$ is $L^2(D)$-valued progressively measurable such that

$$\mathbf{E}\int_0^T ||\sigma(t)||_2^2 dt < \infty, \tag{2.8}$$

Definition 2.1. Under the assumption (2.7) and (2.8), u is said to be a solution of (1.2) on the interval $[0,T]$ if

$$(u, u_t) \text{ is } \mathrm{H}_0^1(\mathrm{D}) \times \mathrm{L}^2(\mathrm{D})\text{−valued progressively measurable}, \tag{2.9}$$

$$(u, u_t) \in L^2(\Omega; C([0,T]; H_0^1(D) \times L^2(D))), \quad u_t \in L^q((0,T)\times D), \quad \text{for almost all } \omega, \tag{2.10}$$

$$u(0) = u_0, \quad u_t(0) = u_1, \tag{2.11}$$

$$u_{tt} - \Delta u + |u_t|^{q-2}u_t = |u|^{p-2}u + \varepsilon\sigma(x,t)\partial_t W(t,x) \tag{2.12}$$

holds in the sense of distributions over $(0,T)\times D$ for almost all ω.

Remark 2.1. (2.10) and (2.12) imply that

$$\begin{aligned}\big(u_t(t), \phi\big) = \big(u_1, \phi\big) - \int_0^t \big(\nabla u, \nabla\phi\big)ds - \int_0^t \big(|u_s|^{q-2}u_s, \phi\big)ds \\ + \int_0^t \big(|u|^{p-2}u, \phi\big)ds + \int_0^t \big(\phi, \varepsilon\sigma(x,s)dW_s\big),\end{aligned} \tag{2.13}$$

for all $t \in [0,T]$ and all $\phi \in H_0^1(D)$. In fact, (2.13) is a conventional form for the definition of solution to stochastic differential equations. Here we say u is a strong solution of the equation (1.2).

3. Existence and uniqueness of solution

In this section, we deal with the local existence and uniqueness of solution for problem (1.2) and prove that the solution of (1.2) is global for $q \geq p$. Let $f(u) = |u|^{p-2}u$. For each $N \geq 1$, define a C^1 function χ_N by

$$\chi_N(x) = \begin{cases} 1, & \text{if } x \leq N, \\ \in (0,1), & \text{if } N < x < N+1, \\ 0, & \text{if } x \geq N+1, \end{cases}$$

and further assume that $||\chi_N'||_\infty \leq 2$. We define

$$f_N(u) = \chi_N(||\nabla u||_2) f(u), \quad u \in H_0^1(D).$$

Then, it follows from (2.3) that

$$||f_N(u) - f_N(v)||_2 \leq C_N ||\nabla u - \nabla v||_2, \quad u,\ v \in H_0^1(D), \tag{3.1}$$

where C_N is a constant dependent only on N. Let $g(x) = |x|^{q-2}x$. For any $\lambda > 0$, let

$$g_\lambda(x) = \frac{1}{\lambda}\big(x - (I + \lambda g)^{-1}(x)\big) = g(I + \lambda g)^{-1}(x), \quad x \in \mathbb{R},$$

where g_λ is the Yosida approximation of the mapping g. Since $g(x)$ satisfies maximal monotone and $g'(x) = (q-2)|x|^{q-2} \geq 0$ for any $x \in \mathbb{R}$, then $g_\lambda \in C^1(\mathbb{R})$ and satisfies (see Pazy [27])

$$0 \leq g_\lambda' \leq \frac{1}{\lambda}, \quad |g_\lambda(x)| \leq |g(x)|, \quad |g_\lambda(x)| \leq \frac{1}{\lambda}|x|, \quad \text{for any } x \in \mathbb{R}. \tag{3.2}$$

Lemma 3.1. *Let $\{\lambda_n\}$ be a sequence of positive numbers, and $\{x_n\}$ be a sequence of real numbers such that $\lambda_n \to 0$ and $x_n \to x$. Then*

$$\lim_{n\to\infty} g_{\lambda_n}(x_n) = g(x).$$

Proof. There is some $L > 0$ such that $|x_n| \leq L$ for all $n \geq 1$. Since $g(x)$ is maximal monotone, let y_n be a unique number such that $y_n + \lambda_n g(y_n) = x_n$, for each $n \geq 1$. Then we have

$$|y_n| \leq |x_n| \leq L, \quad |x_n - y_n| \leq \lambda_n C,$$

for each $n \geq 1$, where $C = \sup_{|z|\leq L} |g(z)|$. Now the above assertion follows from

$$|g(x) - g_{\lambda_n}(x_n)| \leq |g(x) - g(x_n)| + |g(x_n) - g(y_n)|.$$

□

Lemma 3.2. *[See Lemma 1.3 in Lions [28]] Let D be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, $\{\varphi_k\}$, $\varphi \in L^q(D)$, $1 < q < \infty$. If*

$$||\varphi_k||_q \leq C \quad \text{and} \quad \varphi_k(x) \to \varphi(x) \text{ for almost all } x \in D,$$

where C is a constant, then $\varphi_k \to \varphi$ weakly in $L^q(D)$.

In order to obtain the local existence and uniqueness of solution for problem (1.2), we will first establish a lemma for the regularized problem. Fix λ and $N > 0$, we will work on the following initial boundary value problem

$$\begin{cases} u_{tt} - \Delta u + g_\lambda(u_t) = f_N(u) + \varepsilon\sigma(x,t)\partial_t W(t,x), & (x,t) \in D \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial D \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in D, \end{cases} \tag{3.3}$$

where we suppose that

$$(u_0, u_1) \in (H_0^1(D) \cap H^2(D)) \times H_0^1(D) \tag{3.4}$$

and that $\sigma(x,t)$ is $H_0^1(D) \cap L^\infty(D)$-valued progressively measurable such that

$$\mathbf{E}\int_0^T (||\nabla\sigma(t)||_2^2 + ||\sigma(t)||_\infty^2)dt < \infty. \tag{3.5}$$

Lemma 3.3. *Assume (2.1), (3.4) and (3.5) hold. Then there is a pathwise unique solution u of (3.3) such that*

$$u \in L^2\big(\Omega; L^\infty(0,T; H_0^1(D) \cap H^2(D))\big) \cap L^2\big(\Omega; C([0,T]; H_0^1(D))\big)$$

and

$$u_t \in L^2\big(\Omega; L^\infty(0,T; H_0^1(D))\big) \cap L^2\big(\Omega; C([0,T]; L^2(D))\big).$$

Moreover, it holds that

$$\boldsymbol{E}\left(||u_t||^2_{L^\infty(0,T;H_0^1(D))} + ||u||^2_{L^\infty(0,T;H_0^1(D)\cap H^2(D))} + \int_0^T\int_D g_\lambda(u_t)u_t dxdt\right) \le C_N,$$

where C_N denotes a positive constant independent of λ.

Proof. Let

$$u_m(t,x) = \sum_{j=1}^m a_{m,j}(t)e_j(x),$$

where $\{e_j\}_{j=1}^\infty$ is a complete orthonormal base of $H_0^1(D)$ satisfying (2.5) and $a_{m,j}$ form a solution of the following system of stochastic differential equations

$$\begin{cases} a''_{m,j} = -\mu_j a_{m,j} - \Big(g_\lambda\big(\sum_{j=1}^m a'_{m,j}e_j\big), e_j\Big) + \Big(f_N\big(\sum_{j=1}^m a'_{m,j}e_j\big), e_j\Big) + \big(e_j, \varepsilon\sigma(x,t)dW_t\big), \\ a_{m,j}(0) = (u_0, e_j), \quad a'_{m,j}(0) = (u_1, e_j), \end{cases} \tag{3.6}$$

for $1 \le j \le m$. By Itô formula, we have

$$\begin{aligned} ||u'_m(t)||_2^2 + ||\nabla u_m(t)||_2^2 &\le ||u'_m(0)||_2^2 + ||\nabla u_m(0)||_2^2 - 2\int_0^t\int_D g_\lambda\big(u'_m(s)\big)u'_m(s)dxds \\ &+2\int_0^t\int_D f_N(u_m)u'_m dxds + 2\int_0^t (u'_m, \varepsilon\sigma)dW_s + c_0^2 TrR\sum_{j=1}^m\int_0^t |(e_j, \varepsilon\sigma)|^2 ds, \end{aligned} \tag{3.7}$$

and

$$
\begin{aligned}
&||\nabla u_m'(t)||_2^2+||\Delta u_m(t)||_2^2\le ||\nabla u_m'(0)||_2^2+||\Delta u_m(0)||_2^2+2\int_0^t\int_D g_\lambda\big(u_m'(s)\big)\Delta u_m'(s)dxds\\
&-2\int_0^t\int_D f_N\big(u_m(s)\big)\Delta u_m' dxds+2\int_0^t\big(\nabla u_m',\varepsilon\nabla(\sigma dW_s)\big)\\
&+2c_0^2TrR\sum_{j=1}^m\int_0^t|(\nabla e_j,\varepsilon\nabla\sigma)|^2ds+2\sum_{j=1}^m\sum_{i=1}^\infty\lambda_i\int_0^t\big|(e_j,\sigma\nabla e_i)\big|^2ds \qquad (3.8)
\end{aligned}
$$

for all $t\in[0,T]$ and almost all ω, where

$$
TrR=\sum_{i=1}^\infty\lambda_i,\quad c_0:=\sup_{i\ge1}||e_i||_\infty.
$$

From (2.2), (2.3) and (3.2), we get

$$
\int_D f_N\big(u_m\big)u_m'dx\le\int_D\chi_N(||\nabla u_m||_2)|u_m|^{p-1}|u_m'(s)|dx\le C_N||\nabla u_m||_2||u_m'||_2, \qquad (3.9)
$$

$$
\begin{aligned}
-\int_D f_N\big(u_m\big)\Delta u_m'dx&=(p-1)\int_D\chi_N(||\nabla u_m||_2)|u_m|^{p-2}\nabla u_m\cdot\nabla u_m'dx\\
&\le C_N(p-1)|||u_m|^{p-2}\nabla u_m||_2||\nabla u_m'||_2\le C_N||\Delta u_m||_2||\nabla u_m'||_2, \qquad (3.10)
\end{aligned}
$$

and

$$
\int_D g_\lambda\big(u_m'(s)\big)\Delta u_m'(s)dx=-\int_D g_\lambda'\big(u_m'(s)\big)|\nabla u_m'(s)|^2dx\le 0. \qquad (3.11)
$$

By the Burkholder-Davis-Gundy inequality, we have

$$
\begin{aligned}
\mathbf{E}\left(\sup_{t\in[0,T]}\left|\int_0^t\big(u_m'(s),\varepsilon\sigma\big)dW_s\right|\right)&\le C\mathbf{E}\Big(\sup_{t\in[0,T]}||u_m'||_2\Big(\varepsilon^2\sum_{i=1}^\infty\int_0^T\big(\sigma(x,t)Re_i,\sigma(x,t)e_i\big)dt\Big)^{\frac12}\Big)\\
&\le\alpha\mathbf{E}\Big(\sup_{t\in[0,T]}||u_m'||_2^2\Big)+\frac{C\varepsilon^2c_0^2}{\alpha}TrR\mathbf{E}\Big(\int_0^T||\sigma(t)||_2^2dt\Big), \qquad (3.12)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{E}\left(\sup_{t\in[0,T]}\left|\int_0^t\big(\nabla u_m',\nabla(\sigma dW_s)\big)\right|\right)&\le\alpha\mathbf{E}\Big(\sup_{t\in[0,T]}||\nabla u_m'||_2^2\Big)\\
&+\frac{C\varepsilon^2c_0^2}{\alpha}TrR\mathbf{E}\Big(\int_0^T\big(||\nabla\sigma(t)||_2^2+||\sigma(t)||_\infty^2\big)dt\Big). \qquad (3.13)
\end{aligned}
$$

Here and below, C and C_N denote positive constants independent of m and λ. From (3.4), (3.5) and (3.7)–(3.13), by Gronwall's inequality, we have

$$
\mathbf{E}\left(\sup_{t\in[0,T]}||\nabla u_m'||_2^2+\sup_{t\in[0,T]}||u_m||_{H_0^1(D)\cap H^2(D)}+\int_0^T\int_D g_\lambda\big(u_m'(s)\big)u_m'(s)dxds\right)\le C_N. \qquad (3.14)
$$

Define

$$
\mathcal{A}_\lambda=||v||^2_{L^\infty(0,T;H_0^1(D)\cap H^2(D))}+||v'||^2_{L^\infty(0,T;H_0^1(D))}+\int_0^T\int_D g_\lambda\big(v'(s)\big)v'(s)dxds. \qquad (3.15)
$$

It follows from (3.15) that

$$P\Big(\bigcup_{L=1}^{\infty}\bigcap_{j=1}^{\infty}\bigcup_{m=j}^{\infty}\{\mathcal{A}_\lambda(u_m)\le L\}\Big)=1. \tag{3.16}$$

Let $\mathcal{P}_m$ is the orthogonal projection of $L^2(D)$ onto the subspace spanned by $\{e_1,\cdots,e_m\}$, i. e.,

$$\mathcal{P}_m\varphi=\sum_{j=1}^{m}(\varphi,e_j)e_j.$$

From (3.6), we have

$$\partial_t(u'_m-\varepsilon\mathcal{P}_mM(t))=\Delta u_m-\mathcal{P}_mg_\lambda(u'_m)+\mathcal{P}_mf_N(u_m) \tag{3.17}$$

in the sense of distributions over $(0,T)\times D$ for almost all ω, where $M(t)$ is defined by (2.6) with (3.5). Since $\sigma(x,t)$ is $H_0^1(D)\cap L^\infty(D)$-valued progressively measurable and $\{W(t,x):t\ge 0\}$ is a V-valued process, there is a subset $\Omega_1\subset\Omega$ with $P(\Omega\setminus\Omega_1)=0$ such that for each $\omega\in\Omega_1$,

$$M\in C([0,T];H_0^1(D)),\ \text{and (3.17) holds for all m}\ge 1. \tag{3.18}$$

From (3.14), for each $\omega\in\Omega_1$ there is a subsequence $\{u_{m_k}\}_{k=1}^\infty$ such that

$$\mathcal{A}_\lambda(u_{m_k})\le L_\omega,\ \text{for all k}\ge 1\ \text{and for some constant}\ \mathrm{L}_\omega>0, \tag{3.19}$$

$$u_{m_k}\to u\quad\text{weak star in}\ \mathrm{L}^\infty(0,\mathrm{T};\mathrm{H}_0^1(\mathrm{D})\cap\mathrm{H}^2(\mathrm{D})), \tag{3.20}$$

$$u_{m_k}\to u\quad\text{strongly in}\ \mathrm{C}([0,\mathrm{T}];\mathrm{H}_0^1(\mathrm{D})), \tag{3.21}$$

and

$$u'_{m_k}\to u'\quad\text{weak star in}\ \mathrm{L}^\infty(0,\mathrm{T};\mathrm{H}_0^1(\mathrm{D})), \tag{3.22}$$

for some function $u=u(\omega)$. It follows from (3.2) that

$$|g_\lambda(x)|^{\frac{q}{q-1}}\le Cg_\lambda(x)x,\quad\text{for all x}\in\mathbb{R}\ \text{and}\ \lambda>0.$$

From (2.1), we have the embedding $L^{\frac{q}{q-1}}\subset H^{-1}(D)$. Thus, by (3.15) and (3.19), we have

$$||g_\lambda(u'_{m_k})||^{\frac{q}{q-1}}_{L^{\frac{q}{q-1}}(0,T;H^{-1}(D))}\le CL_\omega, \tag{3.23}$$

which combined with (3.17), yields

$$||u'_{m_k}-\varepsilon\mathcal{P}_{m_k}M||_{W^{1,\frac{q}{q-1}}(0,T;H^{-1}(D))}\le CL_\omega \tag{3.24}$$

for all $k\le 1$. By (3.22) and (3.24), we get

$$u'_{m_k}-\varepsilon\mathcal{P}_{m_k}M\to u'-\varepsilon M\quad\text{strongly in}\ \mathrm{C}([0,\mathrm{T}];\mathrm{L}^2(\mathrm{D})). \tag{3.25}$$

This implies that there is a subsequence still denoted by $\{u_{m_k}\}$ such that

$$u'_{m_k}(t,x)\to u'(t,x),\quad\text{for almost all}\ (\mathrm{t},\mathrm{x})\in(0,\mathrm{T})\times\mathrm{D}. \tag{3.26}$$

It follows from (3.23), (3.26) and Lemma 3.2 that

$$g_\lambda(u'_{m_k})\to g_\lambda(u')\quad\text{weakly in}\ \mathrm{L}^{\frac{q}{q-1}}((0,\mathrm{T})\times\mathrm{D}).$$

Thus, $u = u(\omega)$ satisfies (3.3) in the sense of distributions over $(0,T)\times D$. Here the choice of the above subsequence may depend on $\omega \in \Omega_1$. If there is another subsequence which converges to $\widetilde{u} = \widetilde{u}(\omega)$ in the above sense, then $w = u(\omega) - \widetilde{u}(\omega)$ satisfies

$$w'' - \Delta w + g_\lambda(u'(\omega)) - g_\lambda(\widetilde{u}'(\omega)) = f_N(u(\omega)) - f_N(\widetilde{u}(\omega)),$$
$$w(0) = 0, \quad w'(0) = 0,$$
$$w \in L^\infty(0,T; H_0^1(D)\cap H^2(D)) \cap C([0,T); H_0^1(D)),$$
$$w' \in L^\infty(0,T; H_0^1(D)) \cap C([0,T); L^2(D)).$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}(||w'(t)||_2^2 + ||\nabla w(t)||_2^2) + \int_D \big(g_\lambda(u') - g_\lambda(\widetilde{u}')\big)w'dx = \int_D \big(f_N(u) - f_N(\widetilde{u})\big)w'dx. \tag{3.27}$$

From (3.2), we get

$$\int_D g_\lambda(u'(\omega)) - g_\lambda(\widetilde{u}'(\omega))w'dx \geq 0.$$

By the Hölder inequality, it follows from (2.1) that

$$\begin{aligned}
&\left|\int_D (f_N(u) - f_N(\widetilde{u}))w'dx\right| = \left|\int_D \big(\chi_N(||\nabla u||_2)|u|^{p-2}u - \chi_N(||\nabla \widetilde{u}||_2)|\widetilde{u}|^{p-2}\widetilde{u}\big)w'dx\right| \\
&\leq C_N(p-1)\int_D \sup\{|u|^{p-2}, |\widetilde{u}|^{p-2}\}|w||w'|dx \leq C_N(||u||_{(p-2)d}^{(p-2)} + ||\widetilde{u}||_{(p-2)d}^{(p-2)})||w||_{\frac{2d}{d-2}}||w'||_2 \\
&\leq C_N||\nabla w(t)||_2||w'||_2.
\end{aligned} \tag{3.28}$$

Combining with (3.27) with (3.28), we have

$$||w'(t)||_2^2 + ||\nabla w(t)||_2^2 \leq 2C_N\int_0^t (||w'(s)||_2^2 + ||\nabla w(s)||_2^2)ds,$$

which implies $w = 0$, i.e., $u(\omega) = \widetilde{u}(\omega)$. Hence, for each $\omega \in \Omega_1$, $u = u(\omega)$ is well-defined.

We shall also show that (u, u_t) is $(H_0^1(D)\cap H^2(D)) \times H_0^1(D)$-valued progressively measurable for any $0 \leq t \leq T$. Let $\mathbf{B}_r(z)$ be a closed ball in $C([0,T]; H_0^1(D)\times L^2(D))$ with radius $r > 0$ and center at z. Then by virtue of the way u has been obtained, it holds that

$$\{(u,u_t) \in \mathbf{B}_r(z)\} \cap \Omega_1 = \Omega_1 \cap \bigcup_{L=1}^{\infty}\bigcap_{\nu=1}^{\infty}\bigcap_{j=1}^{\infty}\bigcup_{m=j}^{\infty}\left\{\big((u_m, u_m') \in \mathbf{B}_{r+1/\nu}(z)\big) \cap \big(\mathcal{A}_\lambda(u_m) \leq L\big)\right\}. \tag{3.29}$$

Since $(u, u_t) \in C([0,T]; H_0^1(D)\times L^2(D))$ for almost all ω, and the right- hand side of (3.29) belongs to $\mathcal{F}_T$, it holds that

$$\big\{(t,\omega)|0 \leq t \leq T, (u(t,\omega), u_t(t,\omega)) \in A\big\} \in \mathcal{B}([0,T])\otimes\mathcal{F}_T, \tag{3.30}$$

for every $A \in \mathcal{B}(H_0^1(D)\times L^2(D))$. Since every closed ball of finite radius in $(H_0^1(D)\cap H^2(D))\times H_0^1(D)$ is closed in $H_0^1(D)\times L^2(D)$, we have $\mathcal{B}((H_0^1(D)\cap H^2(D))\times H_0^1(D)) \subset \mathcal{B}(H_0^1(D)\times L^2(D))$. Thus, (3.30) holds for every $\mathcal{B}((H_0^1(D)\cap H^2(D))\times H_0^1(D))$. By the pathwise uniqueness, we may replace T in (3.30) by any $0 \leq t \leq T$ and (u, u_t) is $(H_0^1(D)\cap H^2(D))\times H_0^1(D)$-valued progressively measurable.

Next we show that for each $\omega \in \Omega_1$,

$$\mathcal{A}_\lambda(u)\wedge K \leq \underline{\lim}_{m\to\infty}\mathcal{A}_\lambda(u_m)\wedge K \tag{3.31}$$

for each $K>0$. If $\underline{\lim}_{m\to\infty}\mathcal{A}_\lambda(u_m)\wedge K=K$, then the inequality is obvious. If $\underline{\lim}_{m\to\infty}\mathcal{A}_\lambda(u_m)\wedge K=\delta<K$, then there is a subsequence $\{u_{m_k}\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty}\mathcal{A}_\lambda(u_{m_k})=\delta,$$

and $\{u_{m_k}(\omega)\}$ converges to $u(\omega)$ in the sense of (3.19)-(3.22) and (3.25). It follows that

$$||u||_{L^\infty(0,T;H_0^1(D)\cap H^2(D))}\le \underline{\lim}_{k\to\infty}||u_{m_k}||_{L^\infty(0,T;H_0^1(D)\cap H^2(D))},$$
$$||u'||_{L^\infty(0,T;H_0^1(D))}\le \underline{\lim}_{k\to\infty}||u'_{m_k}||_{L^\infty(0,T;H_0^1(D))},$$

and

$$\int_0^T\int_D g_\lambda\big(u'(s)\big)u'(s)dxds\le \underline{\lim}_{k\to\infty}\int_0^T\int_D g_\lambda\big(u'_{m_k}(s)\big)u'_{m_k}(s)dxds,$$

which yield

$$\mathcal{A}_\lambda(u)\le\delta.$$

Thus, (3.31) is valid. By (3.14), (3.31) and Fatou's lemma, we have

$$\mathbf{E}(\mathcal{A}_\lambda(u)\wedge K)\le C_N,$$

for some constant C_N independent of K and λ. By passing $K\uparrow\infty$, we get

$$\mathbf{E}(\mathcal{A}_\lambda(u))\le C_N. \tag{3.32}$$

□

Next we still fix $N>0$ and consider the following equation

$$\begin{cases} u_{tt}-\Delta u+g(u_t)=f_N(u)+\varepsilon\sigma(x,t)\partial_t W(t,x), & (x,t)\in D\times(0,T),\\ u(x,t)=0, & (x,t)\in\partial D\times(0,T),\\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), & x\in D.\end{cases} \tag{3.33}$$

Lemma 3.4. *Assume (2.1), (3.4) and (3.5) hold. Then there is a pathwise unique solution u of (3.33) such that*

$$u\in L^2\big(\Omega;L^\infty(0,T;H_0^1(D)\cap H^2(D))\big)\cap L^2\big(\Omega;C([0,T];H_0^1(D))\big),$$
$$u_t\in L^2\big(\Omega;L^\infty(0,T;H_0^1(D))\big)\cap L^2\big(\Omega;C([0,T];L^2(D))\big),$$

and

$$u_t\in L^q((0,T)\times D).$$

Proof. We denote by u_λ the solution of (3.3) under the conditions (3.4) and (3.5). Since $\mathbf{E}(\mathcal{A}_\lambda(u_\lambda))\le C_N$ for all $\lambda>0$, we can repeat the same argument as above by considering $\lambda=\frac{1}{m}$, $m=1,2,\cdots$. there is $\Omega_2\subset\Omega$ with $P(\Omega\setminus\Omega_2)=0$ and the following properties. For each $\omega\in\Omega_2$,

$$M\in C([0,T];H_0^1(D)),\ \text{and for all}\ \lambda=\frac{1}{\mathrm{m}},\ \mathrm{m}\ge 1, \tag{3.34}$$
$$(u'_\lambda-\varepsilon M(t))'-\Delta u_\lambda+g_\lambda(u'_\lambda)=f_N(u_\lambda)$$

holds in the sense of distributions over $(0,T)\times D$, and there is a subsequence satisfying the following.

$$\mathcal{A}_{\lambda_k}(u_{\lambda_k})\le L_\omega,\ \text{for all k}\ge 1\ \text{and for some constant}\ \mathrm{L}_\omega>0, \tag{3.35}$$

$$u_{\lambda_k}\to u\quad \text{weak star in}\ \mathrm{L}^\infty(0,\mathrm{T};\mathrm{H}_0^1(\mathrm{D})\cap\mathrm{H}^2(\mathrm{D})), \tag{3.36}$$

$$u_{\lambda_k} \to u \quad \text{strongly in } \mathrm{C([0,T];H_0^1(D))}, \tag{3.37}$$

$$u'_{\lambda_k} \to u' \quad \text{weak star in } \mathrm{L^\infty(0,T;H_0^1(D))}, \tag{3.38}$$

$$u'_{\lambda_k} \to u' \quad \text{strongly in } \mathrm{C([0,T];L^2(D))}, \tag{3.39}$$

and

$$u'_{\lambda_k} \to u' \quad \text{for almost all } \mathrm{(x,t) \in (0,T) \times D}, \tag{3.40}$$

for some function $u = u(\omega)$. By Lemma 3.1,

$$g_{\lambda_k}(u'_{\lambda_k}) \to g(u') \quad \text{for almost all } \mathrm{(x,t) \in (0,T) \times D}.$$

It follows from (3.35) and Lemma 3.2 that

$$g_{\lambda_k}(u'_{\lambda_k}) \to g(u') \quad \text{weakly in } \mathrm{L^{\frac{q}{q-1}}((0,T) \times D)}.$$

Thus, $u = u(\omega)$ satisfies (3.33) in the sense of distributions over $(0,T)\times D$ for $\omega \in \Omega$. Suppose that for $\omega \in \Omega$, there is another subsequence which converges to $\widetilde{u} = \widetilde{u}(\omega)$ in the sense of (3.35)-(3.40). Similarly the proof in Lemma 3.3, we can show that $u(\omega) = \widetilde{u}(\omega)$ follows from the equation

$$u_{tt}(\omega) - \widetilde{u}_{tt}(\omega) - \Delta(u(\omega) - \widetilde{u}(\omega)) + g(u_t(\omega)) - g(\widetilde{u}_t(\omega)) = f_N(u(\omega)) - f_N(\widetilde{u}(\omega)),$$

and the regularity

$$u(\omega), \widetilde{u}(\omega) \in L^\infty(0,T; H_0^1(D) \cap H^2(D)) \cap C([0,T); H_0^1(D)),$$

$$u_t(\omega), \widetilde{u}_t(\omega) \in L^\infty(0,T; H_0^1(D)) \cap C([0,T); L^2(D)),$$

$$g(u_t(\omega)), g(\widetilde{u}_t(\omega)) \in L^{\frac{q}{q-1}}((0,T) \times D).$$

Again by the same argument as Lemma 3.3, (u, u_t) is $(H_0^1(D) \cap H^2(D)) \times H_0^1(D)$-valued progressively measurable. Next we define

$$\mathcal{A}(u) = ||u||^2_{L^\infty(0,T;H_0^1(D)\cap H^2(D))} + ||u_t||^2_{L^\infty(0,T;H_0^1(D))} + \int_0^T \int_D g_\lambda(u_t) u_t dx dt. \tag{3.41}$$

Then by the same argument as (3.32), we have

$$\mathbf{E}\big(\mathcal{A}(u)\big) \le C_N. \tag{3.42}$$

□

Now we consider the local existence and uniqueness of solution for problem (1.2) under the assumption (2.7).

Theorem 3.5. *Under the assumptions (2.1), (2.7) and (2.8), there is a pathwise unique local solution u of (1.2) according to Definition 2.1 such that the energy equation holds:*

$$\begin{aligned}&||\nabla u(t)||_2^2 + ||u_t(t)||_2^2 + 2\int_0^t \int_D |u_t(s)|^q dx ds - 2\int_0^t \int_D |u(s)|^{p-2} u(s) u_t(s) dx ds \\ &= ||\nabla u_0||_2^2 + ||u_1||_2^2 + 2\int_0^t \big(u_t(s), \varepsilon\sigma(x,s)\big) dW_s + \varepsilon^2 \sum_{i=1}^\infty \int_0^t \int_D \lambda_i e_i^2(x) \sigma^2(x,s) dx ds.\end{aligned} \tag{3.43}$$

Proof. Let us choose sequences $\{u_{0,m}\}$, $\{u_{1,m}\}$ and $\{\sigma_m(x,t,\omega)\}$ such that

$$u_{0,m} \in H_0^1(D) \cap H^2(D), \quad u_{1,m} \in H_0^1(D), \quad \sigma_m(x,t,\omega) \in L^2(\Omega; L^2(0,T; H_0^1(D) \cap L^\infty(D)))$$

$$\mathbf{E}\int_0^T (||\nabla \sigma_m(t)||_2^2 + ||\sigma_m(t)||_\infty^2)dt < \infty.,$$

and as $m \to \infty$,

$$u_{0,m} \to u_0 \quad \text{strongly in} \quad \mathrm{H}_0^1(\mathrm{D}), \tag{3.44}$$

$$u_{1,m} \to u_1 \quad \text{strongly in} \quad \mathrm{L}^2(\mathrm{D}), \tag{3.45}$$

$$\mathbf{E}\int_0^T ||\sigma_m(x,t) - \sigma(x,t)||_2^2 dt \to 0. \tag{3.46}$$

For each $m \ge 1$, let u_m be the solution of

$$\begin{cases} u_{tt} - \Delta u + g(u_t) = f_N(u) + \varepsilon \sigma_m(x,t)\partial_t W(t,x), & (x,t) \in D \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial D \times (0,T), \\ u(x,0) = u_{0,m}(x), \quad u_t(x,0) = u_{1,m}(x), & x \in D. \end{cases} \tag{3.47}$$

By Lemma 3.4, we have

$$u_m \in L^2\big(\Omega; L^\infty(0,T; H_0^1(D) \cap H^2(D))\big) \cap L^2\big(\Omega; C([0,T]; H_0^1(D))\big), \tag{3.48}$$

$$u_m' \in L^2\big(\Omega; L^\infty(0,T; H_0^1(D))\big) \cap L^2\big(\Omega; C([0,T]; L^2(D))\big), \tag{3.49}$$

and the energy equation

$$\begin{aligned} &||\nabla u_m||_2^2 + ||u_m'||_2^2 + 2\int_0^t \int_D |u_m'|^q dxds - 2\int_0^t\int_D \chi(||\nabla u_m||_2)|u_m|^{p-2}u_m u_m'(s)dxds \\ &= ||\nabla u_{0,m}||_2^2 + ||u_{1,m}||_2^2 + 2\int_0^t (u_m', \varepsilon\sigma_m)dW_s + \varepsilon^2 \sum_{i=1}^\infty \int_0^t\int_D \lambda_i e_i^2(x)\sigma_m^2(x,s)dxds. \end{aligned} \tag{3.50}$$

Let

$$M_m(t,x) = \int_0^t \sigma_m(x,s)dW(s,x), \quad t > 0,\ x \in D.$$

Then, for any m_1, m_2

$$(u_{m_1}'' - u_{m_2}'') - \Delta(u_{m_1} - u_{m_2}) + g(u_{m_1}') - g(u_{m_2}') = f_N(u_{m_1}) - f_N(u_{m_2}) + \varepsilon(M_{m_1} - M_{m_2})' \tag{3.51}$$

holds in the sense of distributions over $(0,T) \times D$ for almost all ω. For the damping term, we use the following elementary inequality

$$(|a|^{q-2}a - |b|^{q-2}b)(a-b) \ge c|a-b|^q \tag{3.52}$$

for $a,\ b \in \mathbb{R}$, $q \ge 2$, where c is a positive constant. By inequality (3.52) and the regularity (3.48) and (3.49), we can drive from

$$\begin{aligned} &||u_{m_1}'(t) - u_{m_2}'(t)||_2^2 + ||\nabla u_{m_1}(t) - \nabla u_{m_2}(t)||_2^2 + 2c\int_0^t ||u_{m_1}' - u_{m_2}'||_q^q ds \\ &\le ||\nabla u_{0,m_1} - \nabla u_{0,m_2}||_2^2 + ||u_{1,m_1} - u_{1,m_2}||_2^2 + 2\int_0^t \big(f_N(u_{m_1}) - f_N(u_{m_2}), u_{m_1}' - u_{m_2}'\big)ds \\ &\quad + 2\varepsilon\int_0^t (\sigma_{m_1} - \sigma_{m_2}, u_{m_1}' - u_{m_2}')dW_s + \varepsilon^2 c_0^2 TrR\int_0^t ||\sigma_{m_1} - \sigma_{m_2}||_2^2 ds \end{aligned} \tag{3.53}$$

for all $t \in [0,T]$. For the third term on the right of (3.53), it follows from (3.1) that

$$
\begin{aligned}
&2\Big|\int_0^t (f_N(u_{m_1}) - f_N(u_{m_2}), u'_{m_1} - u'_{m_2})ds\Big| \le 2\int_0^t ||f_N(u_{m_1}) - f_N(u_{m_2})||_2||u'_{m_1} - u'_{m_2}||_2 ds \\
&\le 2C_N \int_0^t ||\nabla u_{m_1} - \nabla u_{m_2}||_2||u'_{m_1} - u'_{m_2}||_2 ds \\
&\le C_N \int_0^t ||\nabla u_{m_1} - \nabla u_{m_2}||_2^2 dt + C_N \int_0^t ||u'_{m_1} - u'_{m_2}||_2^2 ds, \qquad (3.54)
\end{aligned}
$$

where C_N ia s positive constant independent of m_1 and m_2. In view of (3.53) and (3.54), it follows that

$$
\begin{aligned}
&\mathbf{E}\sup_{0\le t\le T}\Big(||u'_{m_1}(t) - u'_{m_2}(t)||_2^2 + ||\nabla u_{m_1}(t) - \nabla u_{m_2}(t)||_2^2\Big) \\
&\le ||\nabla u_{0,m_1} - \nabla u_{0,m_2}||_2^2 + C_N\int_0^T \mathbf{E}\sup_{0\le t\le T}\Big(||\nabla u_{m_1} - \nabla u_{m_2}||_2^2 + ||u'_{m_1} - u'_{m_2}||_2^2\Big)ds \\
&\quad + ||u_{1,m_1} - u_{1,m_2}||_2^2 + \varepsilon^2 c_0^2 TrR\mathbf{E}\int_0^t ||\sigma_{m_1} - \sigma_{m_2}||_2^2 ds \\
&\quad + 2\varepsilon\mathbf{E}\sup_{0\le t\le T}\Big|\int_0^t (\sigma_{m_1} - \sigma_{m_2}, u'_{m_1} - u'_{m_2})dW_s\Big|. \qquad (3.55)
\end{aligned}
$$

For the last term on the right of (3.55), by the Burkholder-Davis-Gundy inequality we have

$$
\begin{aligned}
&\mathbf{E}\Big(\sup_{t\in[0,T]}\Big|\int_0^t (\sigma_{m_1} - \sigma_{m_2}, u'_{m_1} - u'_{m_2})dW_s\Big|\Big) \\
&\le C\mathbf{E}\Big(\sup_{t\in[0,T]}||u'_{m_1} - u'_{m_2}||_2\Big(\sum_{i=1}^{\infty}\int_0^t \big((\sigma_{m_1} - \sigma_{m_2})Re_i, (\sigma_{m_1} - \sigma_{m_2})e_i\big)dt\Big)^{\frac{1}{2}}\Big) \\
&\le \alpha\mathbf{E}\Big(\sup_{t\in[0,T]}||u'_{m_1} - u'_{m_2}||_2^2\Big) + \frac{Cc_0^2}{\alpha}TrR\mathbf{E}\int_0^t ||\sigma_{m_1} - \sigma_{m_2}||_2^2 dt \qquad (3.56)
\end{aligned}
$$

where α and C are some positive constants. By taking(3.55), (3.56) into account and invoking the Gronwall inequality again, we get

$$
\begin{aligned}
&\mathbf{E}\Big(\sup_{t\in[0,T]}||u'_{m_1} - u'_{m_2}||_2^2 + \sup_{t\in[0,T]}||\nabla u_{m_1} - \nabla u_{m_2}||_2^2\Big) \\
&\le C_N\Big(||\nabla u_{0,m_1} - \nabla u_{0,m_2}||_2^2 + ||u_{1,m_1} - u_{1,m_2}||_2^2 + \varepsilon^2 c_0^2 TrR\mathbf{E}\int_0^t ||\sigma_{m_1} - \sigma_{m_2}||_2^2 ds\Big). \qquad (3.57)
\end{aligned}
$$

Moreover, it can be derived from (3.53) and (3.57) that

$$
\begin{aligned}
&\mathbf{E}\Big(\int_0^T \sup_{t\in[0,T]}||u'_{m_1} - u'_{m_2}||_q^q dt\Big) \\
&\le C_N\Big(||\nabla u_{m_1}(t) - \nabla u_{m_2}(t)||_2^2 + ||u_{1,m_1} - u_{1,m_2}||_2^2 + \varepsilon^2 c_0^2 TrR\mathbf{E}\int_0^t ||\sigma_{m_1} - \sigma_{m_2}||_2^2 ds\Big). \qquad (3.58)
\end{aligned}
$$

It follows from (3.44)-(3.46) and (3.57) that $\{u_m\}$ and $\{u'_m\}$ are cauchy sequences in $L^2(\Omega; H_0^1(D))$ and $L^2(\Omega; L^2(D))$, respectively. Thus,

$$
(u_m, u'_m) \to (u_N, u'_N) \quad \text{strongly in} \quad \mathrm{L^2(\Omega; C([0,T]; H_0^1(D) \times L^2(D)))} \qquad (3.59)
$$

for some function u_N dependent on N. Also, by (3.58), $\{u'_m\}$ are cauchy sequences in $L^q((0,T)\times D)$. So $\{u'_m\}$ converge strongly in $L^q((0,T)\times D)$. Then there exist subsequences of $\{u'_m\}$, still denoted by $\{u'_m\}$ such that

$$u'_m \to u'_N \quad \text{for almost all } (\mathrm{x},\mathrm{t}) \in (0,\mathrm{T}) \times \mathrm{D}. \tag{3.60}$$

It follows from (3.42), (3.60) and Lemma 3.2 that

$$|u'_m|^{q-2}u'_m \to |u'_N|^{q-2}u'_N \quad \text{weakly in } \mathrm{L}^{\frac{q}{q-1}}((0,\mathrm{T})\times \mathrm{D}). \tag{3.61}$$

Therefore, using (3.59), (3.60), the convergence of the initial data and $\sigma_m(x,t)$, u_N is the solution of the following equation

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{q-2}u_t = f_N(u) + \varepsilon\sigma(x,t)\partial_t W(t,x), & (x,t)\in D\times(0,T),\\ u(x,t) = 0, & (x,t)\in \partial D\times(0,T),\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x\in D, \end{cases} \tag{3.62}$$

which satisfies the requirements of Definition 2.1, where u_0, u_1 and $\sigma(x,t)$ satisfy condition (2.7). For uniqueness of (3.62), the proof is similar in the Lemma 3.3, so we omit it here.

To obtain the energy equation of (3.62), we proceed by taking the termwise limit in the approximate equation (3.50). It is ease to show that

$$||\nabla u_m||_2^2 \to ||\nabla u_N||_2^2, \quad ||u'_m||_2^2 \to ||u'_N||_2^2, \quad ||\nabla u_{0,m}||_2^2 \to ||\nabla u_0||_2^2, \quad ||u_{1,m}||_2^2 \to ||u_1||_2^2$$

in the mean and

$$\int_0^t\int_D |u'_m|^q \to \int_0^t\int_D |u'_N|^q$$

by (3.61). By the dominated convergence theorem, the term $\int_0^t\int_D \lambda_1 e_i^2(x)\sigma_m^2(x,s)dxds$ converges in the mean to $\int_0^t\int_D \lambda_1 e_i^2(x)\sigma^2(x,s)dxds$. For the remaining two terms in (3.50), we first consider

$$\begin{aligned} &\Big|\int_0^t\int_D \chi(||\nabla u_m||_2)|u_m|^{p-2}u_m u'_m(s)dxds - \int_0^t\int_D \chi(||\nabla u_N||_2)|u_N|^{p-2}u_N u'_N(s)dxds\Big| \\ &\le \int_0^t \Big|\big(f_N(u_m)-f_N(u_N),u'_N\big)\Big|ds + \int_0^t\Big|\big(f_N(u_m),u'_m-u'_N\big)\Big|ds. \end{aligned} \tag{3.63}$$

From (3.1) and (2.2), we get

$$\Big|\big(f_N(u_m)-f_N(u_N),u'_N\big)\Big| \le ||f_N(u_m)-f_N(u_N)||_2||u'_N||_2 \le C_N||\nabla u_m-\nabla u_N||_2||u'_N||_2, \tag{3.64}$$

and

$$\Big|\big(f_N(u_m),u'_m-u'_N\big)\Big| \le C_N||\nabla u_m||_2||u'_m-u'_N||_2. \tag{3.65}$$

substituting (3.64) and (3.65) into (3.63), we obtain

$$\begin{aligned} &\Big|\int_0^t\int_D \chi(||\nabla u_m||_2)|u_m|^{p-2}u_m u'_m(s)dxds - \int_0^t\int_D \chi(||\nabla u_N||_2)|u_N|^{p-2}u_N u'_N(s)dxds\Big| \\ &\le C_N\int_0^t (||\nabla u_m||_2 + ||u'_N||_2)(||\nabla u_m - \nabla u_N||_2 + ||u'_m-u'_N||_2)ds. \end{aligned}$$

Therefore

$$
\begin{aligned}
&\mathbf{E}\Big|\int_0^t\int_D \chi(||\nabla u_m||_2)|u_m|^{p-2}u_m u_m'(s)dxds-\int_0^t\int_D \chi(||\nabla u_N||_2)|u_N|^{p-2}u_N u_N'(s)dxds\Big|^2\\
&\le 2C_N\Big(\mathbf{E}\int_0^T(||\nabla u_m||_2^2+||u_N'||_2^2)ds\Big)\Big(\mathbf{E}\int_0^T(||\nabla u_m-\nabla u_N||_2+||u_m'-u_N'||_2)ds\Big),
\end{aligned}
$$

which converges to zero as $m\to\infty$. Finally, for the stochastic integral term, we have

$$
\begin{aligned}
&\mathbf{E}\Big|\int_0^t (u_m',\sigma_m)dW_s-\int_0^t (u_N',\sigma)dW_s\Big|\\
&\le \mathbf{E}\Big|\int_0^t (u_m'-u_N',\sigma_m)dW_s\Big|+\mathbf{E}\Big|\int_0^t (u_N',\sigma_m-\sigma)dW_s\Big|.
\end{aligned}
$$

Now, by the Burkholder-Davis-Gundy inequality, we have

$$
\begin{aligned}
&\mathbf{E}\sup_{0\le t\le T}\Big|\int_0^t (u_m'-u_N',\sigma_m)dW_s\Big|\le C\mathbf{E}\Big(\sup_{0\le t\le T}||u_m'-u_N'||_2\Big(\sum_{i=1}^\infty\int_0^T(\sigma_m Re_i,\sigma_m e_i)dt\Big)^{\frac12}\Big)\\
&\le Cc_0TrR\mathbf{E}\Big(\sup_{0\le t\le T}||u_m'-u_N'||_2^2\Big)^{\frac12}\mathbf{E}\Big(\int_0^T||\sigma_m||_2^2dt\Big)^{\frac12}\to 0,\quad \text{as m}\to\infty.
\end{aligned}
$$

Similarly,

$$
\mathbf{E}\sup_{0\le t\le T}\Big|\int_0^t (u_N',\sigma_m-\sigma)dW_s\Big|\le Cc_0TrR\mathbf{E}\Big(\sup_{0\le t\le T}||u_N'||_2^2\Big)^{\frac12}\mathbf{E}\Big(\int_0^T||\sigma_m-\sigma||_2^2dt\Big)^{\frac12},
$$

which also tends to zero as $m\to 0$ by (3.46). There above three inequalities imply that

$$
\int_0^t (u_m',\sigma_m)dW_s\to\int_0^t (u_N',\sigma)dW_s,\quad \text{as m}\to\infty.
$$

Hence, we obtain the energy equation of (3.62)

$$
\begin{aligned}
&||\nabla u_N||_2^2+||u_N'||_2^2+2\int_0^t\int_D|u_N'|^q dxds-2\int_0^t\int_D\chi(||\nabla u_N||_2)|u_N|^{p-2}u_N u_N'(s)dxds\\
&=||\nabla u_0||_2^2+||u_1||_2^2+2\int_0^t(u_N',\varepsilon\sigma)dW_s+\varepsilon^2\sum_{i=1}^\infty\int_0^t\int_D\lambda_1 e_i^2(x)\sigma^2(x,s)dxds. \qquad (3.66)
\end{aligned}
$$

For each N, introduce the stopping time τ_N by

$$
\tau_N=\inf\{t>0;||\nabla u_N||_2\ge N\}.
$$

By the uniqueness of the solution of (3.62), for $t\in[0,\tau_N\wedge T)$, $u(t)=u_N(t)$ is the local solution of (1.2). As τ_N is increasing in N, let $\tau_\infty=\lim_{N\to\infty}\tau_N$. Hence, we construct a unique continuous local solution $u(t)=\lim_{N\to\infty}u_N(t)$ to (1.2) on $[0,T\wedge\tau_\infty)$, which satisfies the requirements of Definition 2.1 and the energy equation (3.43). □

To obtain a global solution, it is necessary to consider the interaction between the damping term $|u_t|^{q-2}u_t$ and the source term $|u|^{p-2}u$ such that a certain energy bound can be established to prevent the unlimited growth. To state the next theorem, we define

$$
e\big(u(t)\big)=||u_t(t)||_2^2+||\nabla u(t)||_2^2+\frac{2}{p}||u||_p^p.
$$

Theorem 3.6. *Suppose (2.1), (2.7) and (2.8) hold. If $q \geq p$, then for any $T > 0$, there is a unique solution u of (1.2) according to Definition 2.1 on the interval $[0, T]$ such that*

$$\boldsymbol{E} \sup_{0\leq t\leq T} e(t) < \infty. \tag{3.67}$$

Proof. For any $T > 0$, we will show that $u_N(t) = u(t \wedge \tau_N) \to u$ a.s. as $N \to \infty$ for any $t \leq T$, so that the local solution becomes a global one. To this end, it suffices to show that $\tau_N \to \infty$ as $N \to \infty$ with probability one.

Recall that, for $t \in [0, \tau_N \wedge T)$, $u(t) = u_N(t) = u(t \wedge \tau_N)$ is the local solution of (1.2). By the Theorem 3.5, the following energy equation holds:

$$\begin{aligned} e\big(u(t \wedge \tau_N)\big) &= e(u_0) + 4\int_0^{t\wedge t_N}\int_D |u|^{p-2}uu_t(s)dxds - 2\int_0^{t\wedge t_N}\int_D |u_t(s)|^q dxds \\ &+2\int_0^{t\wedge t_N}\big(u_t(s), \varepsilon\sigma\big)dW_s + \varepsilon^2\int_0^{t\wedge t_N}\Big(\sigma(x,s)Re_i(x), \sigma(x,s)e_i(x)\Big)ds. \end{aligned} \tag{3.68}$$

Using Hölder inequality and Young's inequality, we get

$$\Big|\int_D |u|^{p-2}uu_t(s)dx\Big| \leq ||u||_p^{p-1}||u_t||_p \leq \beta||u_t||_p^p + C_\beta||u||_p^p, \tag{3.69}$$

where $\beta > 0$ and C_β is a constant depending on β. Since $q \geq p$ and (2.1), the embedding inequality yields

$$||u_t||_p^p \leq C||u_t||_q^p, \tag{3.70}$$

where C is the embedding constant. Therefore, from (3.68), (3.69) and (3.70), we get

$$\begin{aligned} e\big(u(t \wedge \tau_N)\big) &\leq 4C\beta\int_0^{t\wedge t_N}||u_t(s)||_q^p ds - 2\int_0^{t\wedge t_N}||u_t(s)||_q^q dxds + 4C_\beta\int_0^{t\wedge t_N}||u||_p^p ds \\ &+e(u_0) + 2\int_0^{t\wedge t_N}\big(u_t(s), \varepsilon\sigma\big)dW_s + \varepsilon^2 c_0^2 TrR\int_0^{t\wedge t_N}||\sigma(s)||_2^2 ds. \end{aligned} \tag{3.71}$$

Using $q \geq p$, at this point we distinguish two case:

(i) Either $||u_t||_q^q > 1$ so we choose β small such that $-2||u_t||_q^q + 4C\beta||u_t||_q^p \leq 0$.

(ii) Or $||u_t||_q^q \leq 1$, in this case we have $-2||u_t||_q^q + 4C\beta||u_t||_q^p \leq 4C\beta$.

Therefore in either case, we have

$$\begin{aligned} e\big(u(t \wedge \tau_N)\big) &\leq e(u_0) + 4C\beta(t\wedge t_N) + 4C_\beta\int_0^{t\wedge t_N}||u||_p^p ds \\ &+2\int_0^{t\wedge t_N}\big(u_t(s), \varepsilon\sigma\big)dW_s + \varepsilon^2 c_0^2 TrR\int_0^{t\wedge t_N}||\sigma(s)||_2^2 ds. \end{aligned} \tag{3.72}$$

By taking the expectation of (3.72), we obtain

$$\mathbf{E}e\big(u(t \wedge \tau_N)\big) \leq e(u_0) + 4C\beta(t\wedge t_N) + \varepsilon^2 c_0^2 TrR\int_0^{t\wedge t_N}\mathbf{E}||\sigma(s)||_2^2 ds + K\int_0^{t\wedge t_N}\mathbf{E}e\big(u(s)\big)ds,$$

where $K > 0$ is a constant, which, by the Gronwall inequality and (2.7) , implies that

$$\mathbf{E}e\big(u(T \wedge \tau_N)\big) \leq \big(e(u_0) + CT\big)e^{KT} \leq C_T. \tag{3.73}$$

On the other hand, we have

$$\mathbf{E}e\big(u(T \wedge \tau_N)\big) \geq \mathbf{E}\Big(I(\tau_N \leq T)e\big(u(\tau_N)\big)\Big) \geq C\mathbf{E}\Big(||u_{\tau_N}||_2^2 I(\tau_N \leq T)\Big) \geq CN^2P(\tau_N \leq T),$$

where I is the indicator function. In view of (3.73), the above inequality gives

$$P(\tau_\infty \leq T) \leq P(\tau_N \leq T) \leq \frac{C_T}{N^2},$$

which, with the aid of the Borel-cantelli Lemma. implies that

$$P(\tau_\infty \leq T) = 0.$$

or

$$\lim_{N\to\infty} \tau_N = \infty \quad a.s..$$

Hence, on $[0, \tau_\infty \wedge T) = [0, T)$, $u = \lim_{N\to\infty} u_N(t)$ is the global solution as announced. Since $T > 0$ was chosen arbitrarily, we may replace $[0, T)$ by $[0, T]$.

To verify the energy bound (3.67), by the energy equation (3.43), (3.69),(3.70) and (2.7), we have

$$e\big(u(t)\big) \leq e(u_0) + (4C\beta + \varepsilon^2 C_1)t + 4KC_\beta \int_0^t e\big(u(s)\big)ds + 2\int_0^t \big(u_t(s), \varepsilon\sigma\big)dW_s,$$

where C_1 and K are positive constants. The above inequality yields

$$\mathbf{E}\sup_{0\leq t\leq T} e\big(u(t)\big) \leq e(u_0) + (4C\beta + \varepsilon^2 C_1)T + 4KC_\beta \int_0^T \mathbf{E}\sup_{0\leq s\leq T} e\big(u\big)ds + 2\mathbf{E}\sup_{0\leq t\leq T}\int_0^t \big(u_t, \varepsilon\sigma\big)dW_s. \tag{3.74}$$

By the Burkholder-Davis-Gundy inequality, we have

$$\mathbf{E}\sup_{0\leq t\leq T}\Big|\int_0^t \big(u_t, \varepsilon\sigma\big)dW_s\Big| \leq C_2\mathbf{E}\Big(\sup_{0\leq t\leq T}||u_t||_2\Big(\varepsilon^2\sum_{i=1}^\infty \int_0^T \big(\sigma Re_i, \sigma e_i\big)dt\Big)^{\frac{1}{2}}\Big) \tag{3.75}$$

$$\leq \frac{1}{4}\mathbf{E}\sup_{0\leq t\leq T}||u_t||_2^2 + C_3 c_0^2\varepsilon^2 TrR\int_0^T \mathbf{E}||\sigma(t)||_2^2 dt \tag{3.76}$$

for some constant C_2, $C_3 > 0$. In view of (2.7), (3.74) and (3.75), there exist positive constants C_4 and C_5 depending on β, T etc. such that

$$\mathbf{E}\sup_{0\leq t\leq T} e\big(u(t)\big) \leq C_4 + C_5\int_0^T \mathbf{E}\sup_{0\leq s\leq T} e\big(u\big)ds.$$

By applying the Gronwall inequality, the above gives

$$\mathbf{E}\sup_{0\leq t\leq T} e\big(u(t)\big) \leq C_4 e^{C_5 T},$$

which implies the energy bound (3.67). □

4. Explosive solution of (1.2)

In this section, we switch to discuss the explosion of the solution to (1.2) for $p > q$. Throughout this section, we suppose that $\sigma(x, t, \omega) \equiv \sigma(x, t)$ such that

$$\int_0^\infty \int_D \sigma^2(x,t)dxdt < \infty. \tag{4.1}$$

As well-known, equation (1.2) is equivalent to the following Itô system

$$\begin{cases} du_t = v_t dt, \\ dv_t = \Big(\Delta u_t - |v_t|^{q-2} v_t + |u_t|^{p-2} u_t\Big) dt + \varepsilon\sigma(x,t) dW(t,x), \\ u_t(x,t) = 0, \quad x \in \partial D, \\ u_0(x,0) = u_0(x), \quad v_0(x,0) = u_1(x), \end{cases} \tag{4.2}$$

where $(u_0, u_1) \in H_0^1(D) \times L^2(D)$. Define energy functional $E(t)$ associated to our system

$$\mathcal{E}(t) = \frac{1}{2}||v_t(t)||_2^2 + \frac{1}{2}||\nabla u_t(t)||_2^2 - \frac{1}{p}||u_t||_p^p.$$

Before we state and prove our explosion result, we need the following lemmas.

Lemma 4.1. *Assume (2.1) and (4.1) hold. Let (u_t, v_t) be a solution of system (4.2) with initial data $(u_0, u_1) \in H_0^1(D) \times L^2(D)$. Then we have*

$$\frac{d}{dt}\boldsymbol{E}\mathcal{E}(t) = -\boldsymbol{E}||v_t||_q^q + \frac{1}{2}\varepsilon^2 \sum_{i=1}^{\infty} \int_D \lambda_i e_i^2(x)\sigma^2(x,t)dx, \tag{4.3}$$

where $r(x,x)$ is defined in Section 2, and

$$\begin{aligned} \boldsymbol{E}\big(u_t(t), v_t(t)\big) = \big(u_0(x), v_0(x)\big) - \int_0^t \boldsymbol{E}||\nabla u_s||_2^2 ds + \int_0^t \boldsymbol{E}||v_s(s)||_2^2 ds \\ - \int_0^t \boldsymbol{E}\big(|v_s|^{q-2}v_s, u_s\big)ds + \int_0^t \boldsymbol{E}||u_s(s)||_p^p ds, \end{aligned} \tag{4.4}$$

Proof. Using Itô formula to $||v_t||_2^2$, we have

$$\begin{aligned} ||v_t||_2^2 &= ||v_0||_2^2 + 2\int_0^t (v_s, dv_s) + \int_0^t (dv_s, dv_s) \\ &= ||v_0||_2^2 - 2\int_0^t (\nabla u_s, \nabla v_s)ds - 2\int_0^t ||v_s||_q^q ds + 2\int_0^t (v_s, |u_s|^{p-2}u_s)ds \\ &\quad + 2\int_0^t (v_s, \varepsilon\sigma(x,s)dW(s)) + \varepsilon^2 \sum_{i=1}^{\infty} \int_0^t \big(\sigma(x,s)Re_i, \sigma(x,s)Re_i\big)ds \\ &= 2\mathcal{E}(0) - ||\nabla u_t(t)||_2^2 - 2\int_0^t ||v_s||_q^q ds + \frac{2}{p}||u_t(t)||_p^p \\ &\quad + 2\int_0^t (v_s, \varepsilon\sigma(x,s)dW(s)) + \varepsilon^2 \sum_{i=1}^{\infty} \int_0^t \int_D \lambda_i e_i^2(x)\sigma^2(x,s)dxds. \end{aligned} \tag{4.5}$$

(4.3) follows from (4.5) taking the expectation and taking derivative. Next we turn to prove (4.4).

$$\begin{aligned} \big(u_t(t), v_t(t)\big) &= (u_0, v_0) + \int_0^t \big(u_s(s), dv_s(s)\big) + \int_0^t \big(v_s(s), du_s(s)\big) \\ &= (u_0, v_0) - \int_0^t ||\nabla u_s(s)||_2^2 ds - \int_0^t \big(|v_s|^{q-2}v_s, u_s(s)\big)d\tau \\ &\quad + \int_0^t (u_s, |u_s|^{p-2}u_s)ds + \int_0^t \big(u_s(s), \varepsilon\sigma(x,s)dW(s)\big) + \int_0^t ||v_s(s)||_2^2 ds. \end{aligned} \tag{4.6}$$

Then (4.4) follows from (4.6). $\square$

Let

$$F(t)=\frac{1}{2}\varepsilon^2\sum_{i=1}^{\infty}\int_0^t\int_D\lambda_i e_i^2(x)\sigma^2(x,s)dxds.$$

From (4.1), we have

$$F(\infty)=\frac{1}{2}\varepsilon^2\sum_{i=1}^{\infty}\int_0^{\infty}\int_D\lambda_i e_i^2(x)\sigma^2(x,s)dxds\le\frac{1}{2}\varepsilon^2c_0^2TrR\int_0^{\infty}\int_D\sigma^2(x,s)dxds=E_1<\infty. \quad (4.7)$$

Denote

$$H(t)=F(t)-\mathbf{E}\mathcal{E}(t).$$

Then, by (4.3), we get

$$H'(t)=F'(t)-\frac{d}{dt}\mathbf{E}\mathcal{E}(t)=\mathbf{E}||v_t||_q^q\ge 0. \quad (4.8)$$

Lemma 4.2. *Let (u_t,u_t) is a solution of (4.2). Assume (2.1) holds. Then there exists a positive constant $C>1$ such that*

$$\boldsymbol{E}||u_t||_p^s\le C(F(t)-H(t)-\boldsymbol{E}||v_t||_2^2+\boldsymbol{E}||u_t||_p^p) \quad (4.9)$$

for any $2\le s\le p$.

Proof. If $||u_t||_p^p\le 1$ then $||u_t||_p^s\le||u_t||_p^2\le C||\nabla u_t||_2^2$ by Sobolev embedding. If $||u||_p^p\ge 1$ then $||u_t||_p^s\le||u_t||_p^p$. Therefore, it follows that

$$\mathbf{E}||u_t||_p^s\le C(\mathbf{E}||\nabla u_t||_2^2+\mathbf{E}||u_t||_p^p). \quad (4.10)$$

By the definition of energy function, we have

$$\frac{1}{2}\mathbf{E}||\nabla u_t||_2^2=\mathbf{E}\mathcal{E}(t)-\frac{1}{2}\mathbf{E}||v_t||_2^2+\frac{1}{p}\mathbf{E}||u_t||_p^p=F(t)-H(t)-\frac{1}{2}\mathbf{E}||v_t||_2^2+\frac{1}{p}\mathbf{E}||u_t||_p^p. \quad (4.11)$$

Then, (4.9) follows (4.10) and (4.11). □

In the following, we switch to discuss the explosion of the solution to (1.2) for $p>q$. Actually, we have

Theorem 4.3. *Assume (2.1) and (4.1) hold. Let (u_t,v_t) be the solution of (4.2) with initial data $(u_0,u_1)\in H_0^1(D)\times L^2(D)$ satisfying*

$$\mathcal{E}(0)\le-(1+\beta)E_1, \quad (4.12)$$

where $\beta>0$ is any constant and E_1 is defined in (4.7). If $p>q$, then the solution (u_t,v_t) and the lifespan τ_∞ defined in Section 3 with L^2 norm, either
(1) $\boldsymbol{P}(\tau_\infty<\infty)>0$, i.e., $u_t(t)$ in L^2 norm blows up in finite time with positive probability, or
(2) there existence a positive time $T^*\in(0,T_0]$ such that

$$\lim_{t\to T^*}\boldsymbol{E}\mathcal{E}(t)=+\infty.$$

with

$$T_0=\frac{1-\alpha}{\alpha K\mathcal{E}^{\frac{\alpha}{1-\alpha}}(0)},$$

where α, K are given later.

Proof. For the lifespan τ_∞ of the solution $\{u_t(t);\ t \geq 0\}$ of (1.2) with L^2 norm, let us consider the case when $\mathbf{P}(\tau_\infty = +\infty) = 1$. Then, for sufficiently large $T > 0$, by (4.8) and (4.12), we have

$$0 < (1+\beta)E_1 \leq -\mathcal{E}(0) = H(0) \leq H(t) \leq F(t) + \frac{1}{p}\mathbf{E}||u||_p^p \leq E_1 + \frac{1}{p}\mathbf{E}||u||_p^p. \tag{4.13}$$

Define by

$$L(t) := H^{1-\alpha}(t) + \mu\mathbf{E}(u_t, v_t),$$

for small μ to be chosen later and for

$$0 < \alpha < \min\Big\{\frac{1}{2}, \frac{p-q}{pq}\Big\}. \tag{4.14}$$

Taking a derivative of $L(t)$ and using (4.4) and (4.8), we obtain

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \mu\Big(-\mathbf{E}||\nabla u_t||_2^2 - \mathbf{E}(|v_t|^{q-2}v_t, u_t) + \mathbf{E}||u_t||_p^p + \mathbf{E}||v_t||_2^2\Big) \\ &= (1-\alpha)H^{-\alpha}(t)\mathbf{E}||v_t||_q^q + \mu pH(t) + \mu(\frac{p}{2}+1)\mathbf{E}||v_t||_2^2 \\ &\quad +\mu(\frac{p}{2}-1)\mathbf{E}||\nabla u_t||_2^2 - \mu\mathbf{E}(|v_t|^{q-2}v_t, u_t) - \mu pF(t). \end{aligned} \tag{4.15}$$

Exploiting the inequality $\mathbf{E}||u_t||_q^q \leq C\mathbf{E}||u_t||_p^q$ and the assumption $q < p$, we obtain

$$\begin{aligned} \Big|\mathbf{E}(|v_t|^{q-2}v_t, u_t)\Big| &\leq \big(\mathbf{E}||v_t||_q^q\big)^{\frac{q-1}{q}}\big(\mathbf{E}||u_t||_q^q\big)^{\frac{1}{q}} \leq C\big(\mathbf{E}||v_t||_q^q\big)^{\frac{q-1}{q}}\big(\mathbf{E}||u_t||_p^q\big)^{\frac{1}{q}} \\ &\leq C\big(\mathbf{E}||v_t||_q^q\big)^{\frac{q-1}{q}}\big(\mathbf{E}||u_t||_p^p\big)^{\frac{1}{p}} \leq C\big(\mathbf{E}||v_t||_q^q\big)^{\frac{q-1}{q}}\big(\mathbf{E}||u_t||_p^p\big)^{\frac{1}{q}}\big(\mathbf{E}||u_t||_p^p\big)^{\frac{1}{p}-\frac{1}{q}}. \end{aligned} \tag{4.16}$$

The Young's inequality gives

$$\big(\mathbf{E}||v_t||_q^q\big)^{\frac{q-1}{q}}\big(\mathbf{E}||u_t||_p^p\big)^{\frac{1}{q}} \leq \frac{q-1}{q}k\mathbf{E}||v_t||_q^q + \frac{k^{1-q}}{q}\mathbf{E}||u_t||_p^p. \tag{4.17}$$

In view of (4.13), we get

$$\mathbf{E}||u_t||_p^p \geq p\big(H(t) - F(t)\big) \geq \kappa H(t),$$

where $\kappa = p\beta/(1+\beta)$. We choose α satisfying (4.14) and assume $H(0) > 1$, we have

$$\big(\mathbf{E}||u_t||_p^p\big)^{\frac{1}{p}-\frac{1}{q}} \leq \kappa^{\frac{1}{p}-\frac{1}{q}}H^{\frac{1}{p}-\frac{1}{q}}(t) \leq \kappa^{\frac{1}{p}-\frac{1}{q}}H^{-\alpha}(t) \leq \kappa^{\frac{1}{p}-\frac{1}{q}}H^{-\alpha}(0). \tag{4.18}$$

Substituting (4.17) and (4.18) into (4.16), we obtain

$$\Big|\mathbf{E}(|v_t|^{q-2}v_t, u_t)\Big| \leq C_1\frac{q-1}{q}k\mathbf{E}||v_t||_q^qH^{-\alpha}(t) + C_1\frac{k^{1-q}}{q}\mathbf{E}||u_t||_p^pH^{-\alpha}(0), \tag{4.19}$$

where $C_1 = C\kappa^{\frac{1}{p}-\frac{1}{q}}$. Thus, from (4.15) and (4.19) it follow that

$$\begin{aligned} L'(t) &\geq \Big((1-\alpha) - C_1\frac{q-1}{q}\mu k\Big)H^{-\alpha}(t)\mathbf{E}||v_t||_q^q + \mu pH(t) + \mu(\frac{p}{2}+1)\mathbf{E}||v_t||_2^2 - \mu pF(t) \\ &\quad +\mu(\frac{p}{2}-1)\mathbf{E}||\nabla u_t||_2^2 - \mu C_1\frac{k^{1-q}}{q}H^{-\alpha}(0)\mathbf{E}||u_t||_p^p. \end{aligned} \tag{4.20}$$

We now use Lemma 4.2 with $s = p$ to deduce from (4.20)

$$\begin{aligned} L'(t) &\geq \Big((1-\alpha) - C_1\frac{q-1}{q}\mu k\Big)H^{-\alpha}(t)\mathbf{E}||v_t||_q^q + \mu p H(t) + \mu(\frac{p}{2}+1)\mathbf{E}||v_t||_2^2 - \mu p F(t) \\ &\quad + \mu(\frac{p}{2}-1)\mathbf{E}||\nabla u_t||_2^2 - \mu k^{1-q}C_2\Big(F(t) - H(t) - \mathbf{E}||v_t||_2^2 + \mathbf{E}||u_t||_p^p\Big) \\ &\geq \Big((1-\alpha) - C_1\frac{q-1}{q}\mu k\Big)H^{-\alpha}(t)\mathbf{E}||v_t||_q^q + \mu(\frac{p}{2}+1+k^{1-q}C_2)\mathbf{E}||v_t||_2^2 + \mu(\frac{p}{2}-1)\mathbf{E}||\nabla u_t||_2^2 \\ &\quad + \mu(p + k^{1-q}C_2)H(t) - \mu k^{1-q}C_2\mathbf{E}||u_t||_p^p - \mu(p+k^{1-q}C_2)F(t), \end{aligned} \tag{4.21}$$

where $C_2 = C_1 H^{-\alpha}(0)/q$. Noting that

$$H(t) = F(t) + \frac{1}{p}\mathbf{E}||u_t||_p^p - \frac{1}{2}\mathbf{E}||\nabla u_t||_2^2 - \frac{1}{2}\mathbf{E}||v_t||_2^2$$

and writing $p = 2C_3 + (p - 2C_3)$, where $C_3 < (p-2)/2$, the estimate (4.21) implies

$$\begin{aligned} L'(t) &\geq \Big((1-\alpha) - C_1\frac{q-1}{q}\mu k\Big)H^{-\alpha}(t)\mathbf{E}||v_t||_q^q + \mu(\frac{p}{2}+1+k^{1-q}C_2 - C_3)\mathbf{E}||v_t||_2^2 \\ &\quad + \mu(\frac{p}{2}-1-C_3)\mathbf{E}||\nabla u_t||_2^2 + \mu(p - 2C_3 + k^{1-q}C_2)H(t) \\ &\quad + \mu(\frac{2C_3}{p} - k^{1-q}C_2)\mathbf{E}||u_t||_p^p - \mu(p - 2C_3 + k^{1-q}C_2)F(t). \end{aligned} \tag{4.22}$$

In view of (4.12) and (4.13), we get

$$(p - 2C_3 + k^{1-q}C_2)F(t) \leq (p - 2C_3 + k^{1-q}C_2)E_1 \leq \frac{(p - 2C_3 + k^{1-q}C_2)}{1+\beta}H(t).$$

Substituting the above inequality into (4.22), we get

$$\begin{aligned} L'(t) &\geq \Big((1-\alpha) - C_1\frac{q-1}{q}\mu k\Big)H^{-\alpha}(t)\mathbf{E}||v_t||_q^q + \mu\Big((\frac{p}{2}+1+k^{1-q}C_2 - C_3)\mathbf{E}||v_t||_2^2 \\ &\quad + (\frac{p}{2}-1-C_3)\mathbf{E}||\nabla u_t||_2^2 + (p-2C_3+k^{1-q}C_2)\frac{\beta}{1+\beta}H(t) + (\frac{2C_3}{p} - k^{1-q}C_2)\mathbf{E}||u_t||_p^p\Big). \end{aligned}$$

At this point, we choose k large enough so that the above inequality becomes

$$L'(t) \geq \Big((1-\alpha) - C_1\frac{q-1}{q}\mu k\Big)H^{-\alpha}(t)\mathbf{E}||v_t||_q^q + \mu\gamma\big(H(t) + \mathbf{E}||\nabla u_t||_2^2 + \mathbf{E}||v_t||_2^2 + \mathbf{E}||u_t||_p^p\big), \tag{4.23}$$

where $\gamma > 0$ is the minimum of the coefficients of $H(t)$, $\mathbf{E}||\nabla u_t||_2^2$, $\mathbf{E}||v_t||_2^2$, $\mathbf{E}||u_t||_p^p$ in (4.23). Once k is fixed, we pick μ small enough so that

$$(1-\alpha) - C_1\frac{q-1}{q}\mu k \geq 0$$

and

$$L(0) = H^{1-\alpha}(0) + \mu(u_0, u_1) > 0.$$

Therefore, (4.23) takes on the form

$$L'(t) \geq \mu\gamma\big(H(t) + \mathbf{E}||\nabla u_t||_2^2 + \mathbf{E}||v_t||_2^2 + \mathbf{E}||u_t||_p^p\big) \geq 0. \tag{4.24}$$

Consequently, we have

$$L(t) \geq L(0) > 0, \quad \forall t \geq 0.$$

By Hölder inequality, we get

$$\Big|\mathbf{E}\big(u_t, v_t\big)\Big| \le \big(\mathbf{E}||u_t||_2^2\big)^{\frac{1}{2}}\big(\mathbf{E}||v_t||_2^2\big)^{\frac{1}{2}} \le C\big(\mathbf{E}||u_t||_p^2\big)^{\frac{1}{2}}\big(\mathbf{E}||v_t||_2^2\big)^{\frac{1}{2}},$$

which, by young's inequality implies

$$\begin{aligned}\Big|\mathbf{E}\big(u_t, v_t\big)\Big|^{\frac{1}{1-\alpha}} &\le C\big(\mathbf{E}||u_t||_p^2\big)^{\frac{1}{2(1-\alpha)}}\big(\mathbf{E}||v_t||_2^2\big)^{\frac{1}{2(1-\alpha)}} \\ &\le C\Big(\big(\mathbf{E}||u_t||_p^2\big)^{\frac{\theta}{2(1-\alpha)}} + \big(\mathbf{E}||v_t||_2^2\big)^{\frac{\eta}{2(1-\alpha)}}\Big), \qquad (4.25)\end{aligned}$$

for $1/\theta + 1/\eta = 1$. We take $\eta = 2(1-\alpha)$. Then, by (4.14),

$$\frac{\theta}{2(1-\alpha)} = \frac{1}{1-2\alpha} = \frac{pq}{pq-2p+2q} \le \frac{p}{2},$$

i.e., $2/(1-2\alpha) \le p$. Using $\alpha < 1/2$, (4.25) becomes

$$\Big|\mathbf{E}\big(u_t, v_t\big)\Big|^{\frac{1}{1-\alpha}} \le C\Big(\big(\mathbf{E}||u_t||_p^2\big)^{\frac{1}{1-2\alpha}} + \mathbf{E}||v_t||_2^2\Big) \le C\Big(\mathbf{E}||u_t||_p^{\frac{2}{1-2\alpha}} + \mathbf{E}||v_t||_2^2\Big).$$

Using Lemma 4.2 with $s = 2/(1-2\alpha)$, we get

$$\Big|\mathbf{E}\big(u_t, v_t\big)\Big|^{\frac{1}{1-\alpha}} \le C\big(H(t) + \mathbf{E}||\nabla u_t||_2^2 + \mathbf{E}||v_t||_2^2 + \mathbf{E}||u_t||_p^p\big), \quad \forall t \ge 0. \qquad (4.26)$$

Therefore, we have

$$\begin{aligned}L^{\frac{1}{1-\alpha}}(t) = \Big(H^{1-\alpha}(t) + \mu\mathbf{E}\big(u_t, v_t\big)\Big)^{\frac{1}{1-\alpha}} &\le 2^{\frac{1}{1-\alpha}}\Big(H(t) + \mu\Big|\mathbf{E}\big(u_t, v_t\big)\Big|^{\frac{1}{1-\alpha}}\Big) \\ &\le C\big(H(t) + \mathbf{E}||\nabla u_t||_2^2 + \mathbf{E}||v_t||_2^2 + \mathbf{E}||u_t||_p^p\big), \quad \forall t \ge 0. \qquad (4.27)\end{aligned}$$

Combining (4.24) and (4.27), we obtain

$$L'(t) \ge KL^{\frac{1}{1-\alpha}}, \quad \forall t \ge 0, \qquad (4.28)$$

where K is a positive constant. A simple integration of (4.28) over $(0, t)$ then yields

$$L^{\frac{\alpha}{1-\alpha}}(t) \ge \frac{1-\alpha}{(1-\alpha)L^{-\frac{\alpha}{1-\alpha}}(0) - \alpha K t}. \qquad (4.29)$$

Let

$$T_0 = \frac{1-\alpha}{\alpha K \mathcal{E}^{\frac{\alpha}{1-\alpha}}(0)}.$$

Then $L(t) \to +\infty$ as $t \to T_0$. This means that there exists a positive time $T^* \in (0, T_0]$ such that

$$\lim_{t\to T^*} \mathbf{E}\mathcal{E}(t) = +\infty.$$

As for the case when $\mathbf{P}(\tau_\infty = +\infty) < 1$ (i.e., $\mathbf{P}(\tau_\infty < +\infty) > 0$), then $u_t(t)$ in L^2 norm blows up in finite time interval $[0, \tau_\infty]$ with positive probability. □

Remark 4.1. In the classical (deterministic) case of $\varepsilon = 0$, it is well known that for $(u_0, v_0) \in H_0^1(D) \times L^2(D)$, the condition $\mathcal{E}(0) \le 0$ already imply finite-time blowup of (1.2) (see e.g. [8]). If $\varepsilon > 0$, by our results, to balance the influence of $W(t, x)$ such that the local solution of (1.2) is blow-up with positive probability or explosive in L^2 sense, the initial energy should be satisfied $\mathcal{E}(0) \le -\frac{1}{2}(1+\beta)\varepsilon^2 r_0^2 \int_0^\infty \int_D \sigma^2(x,t)dxdt$.

References

[1] A. Haraux, E. Zuazua, *Decay estimates for some semilinear damped hyperbolic problems*, Arch. Ration. Mech. Anal. **150** (1988), 191–206.

[2] M. Kopackova, *Remarks on bounded solutions of a semilinear dissipative hyperbolic equation*, Comment. Math. Univ. Carolin. **30**(4) (1989), 713–719.

[3] J. Ball, *Remarks on blow up and nonexistence theorems for nonlinear evolutions equations*, Quart. J. Math. Oxford **28** (2) (1977), 473–486.

[4] V.K. Kalantarov, O.A. Ladyzhenskaya, *The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type*, J. Soviet Math. **10** (1978), 53–70.

[5] H.A. Levine, *Instability and nonexistence of global solutions to nonlinear wave equations of the form*, Trans. Amer. Math. Soc. **192** (1974), 1–21.

[6] H.A. Levine, *Some additional remarks on the nonexistence of global solutions to nonlinear wave equations*, SIAM J. Math. Anal. **5** (1974), 138–146.

[7] V. Georgiev, G. Todorova, *Existence of a solution of the wave equation with nonlinear damping and source term*, J. Differential Equations **109** (1994), 295–308.

[8] S.A. Messaoudi, *Blow up in a nonlinearly damped wave equation*, Math. Nachr. **231** (2001), 1–7.

[9] H.A. Levine, J. Serrin, *Global nonexistence theorems for quasilinear evolution equation with dissipation*, Arch. Ration. Mech. Anal. **137** (1997), 341–361.

[10] H.A. Levine, S. Ro Park, *Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation*, J. Math. Anal. Appl. **228** (1998), 181–205.

[11] E. Vitillaro, *Global nonexistence theorems for a class of evolution equations with dissipation*, Arch. Ration. Mech. Anal. **149** (1999), 155–182.

[12] S.A. Messaoudi, B. Said-Houari, *Blow up of solutions of a class of wave equations with nonlinear damping and source terms*, Math. Methods Appl. Sci. **27** (2004), 1687–1696.

[13] P.L. Chow, *Stochastic wave equations with polynomial nonlinearity*, Ann. Appl. Probab. **12** (2002), 361–381.

[14] P.L. Chow, *Nonlinear stochstic wave equations: blow-up of second moments in L^2-norm*, Ann. Appl. Probab. **19** (2009), 2039–2046.

[15] L.J. Bo, D. Tang, Y.G. Wang, *Explosive solutions of stochastic wave equations with damping on $\mathbb{R}^d$*, J. Differential Equations **244** (2008), 170–187.

[16] P.L. Chow, *Asymptotics of solutions to semilinear stochastic wave equations*, Ann. Appl. Probab. **16** (2006), 757–789.

[17] P.L. Chow, *Asymptotic solutions of a nonlinear stochastic beam equation*, Discrete Contin. Dyn. Syst. Ser. B. **6** (2006), 735–749.

[18] Z. Brzeźniak, B. Maslowski, J. Seidler, *Stochastic nonlinear beam equations*, Probab. Theory Related Fields **132** (2005), 119–149.

[19] R. Carmona, D. Nualart, *Random non-linear wave equation: Smoothness of solutions*, Probab. Theory Related Fields **95** (1993), 87–102.

[20] H. Crauel, A. Debussche, F. Flandoli, *Random attractors*, J. Dynam. Differential Equations **9** (1997), 307–341.

[21] R. Dalang, N. Frangos, *The stochastic wave equation in two spatial dimensions*, Ann. Probab. **26** (I) (1998), 187–212.

[22] A. Millet, P.L. Morien, *On a nonlinear stochastic wave equation in the plane: Existence and uniqueness of the solution*, Ann. Appl. Probab. **11** (2001), 922–951.

[23] E. Pardoux, *Equations aux derivées partielles stochastiques nonlinéaries monotones*, Thèse, Université Paris XI 1975.

[24] J.U. Kim, *On the stochastic wave equation with nonlinear damping*, Appl. Math. Optim. **58** (2008), 29–67.

[25] V. Barbu, G.D. Prato, L. Tubaro, *Stochastic wave equations with dissipative damping*, Stochastic Process. Appl. **117** (2007), 1001–1013.

[26] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions, Cambridge University Press*, Cambridge, 1992.

[27] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-verlag, New York, 1983.

[28] J.L. Lions *Quelques méthodes de résolution des problémes aux limites non linéaires*, Dunod, Paris (1969).

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